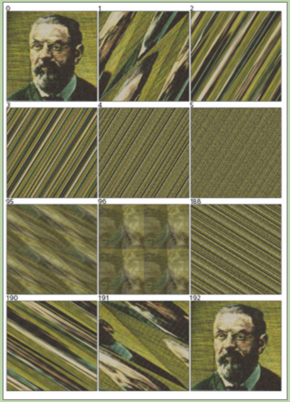
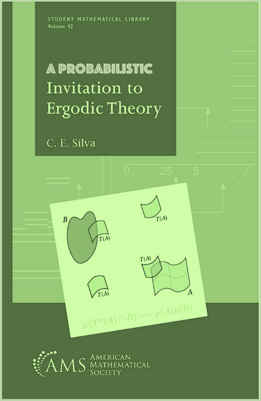
A probabilistic invitation to ergodic theory





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Mathematics and Statistics Colloquium
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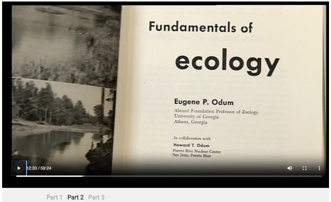
Do systems converge to equilibrium? Are they stable?

Adam Curtis' *All Watched Over By Machines of Loving Grace* (2011) documents the influence of the mathematical theory of dynamical systems on the natural and social sciences and society, especially Norbert Wiener's *Cybernetics* in Part 2 "The Use and Abuse of Vegetational Concepts".

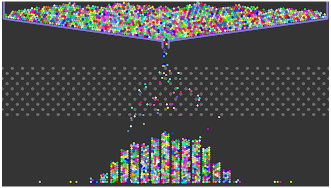


Ergodic theory was invented to answer the questions above. Today, I will:

1. Work up to a precise definition of a dynamical system in equilibrium
2. State theorems relating in equilibrium and out of equilibrium dynamics
3. Argue that some subtleties in ergodic theory are faithfully represented in the simplest example: iterated transposition of two distinct elements.
4. Mention work in progress with Robert Chang (New College of Florida) on central limit theorems (CLTs) in Hamiltonian dynamics with Gaussian random initial data from my Spring 2025 sabbatical spent with Tamara Grava (S.I.S.S.A.) and Peter Miller (U. Michigan)

Reversible dynamics and the emergence of irreversibility

Problem: Predict that typical pile of particles $\approx$ normal Gaussian curve.



Tactic #1: model dynamics and collisions as a dynamical system as in
MATH 322 *Ordinary Differential Equations*
Key assumption: conservation of energy (Hamiltonian)
Key mechanism: **local instabilities** have global consequences

Tactic #2: model dynamics and collisions as a stochastic process as in
MATH 391 *Probability* ... now MATH/STAT 291 *Probability!*
Key assumption: collision outcomes are random according to P
Key mechanism: **local independence** has global consequences

Tactic $\#\frac{3}{2}$: study deterministic dynamical systems with random initial data

"OBSERVABLES" ARE "RANDOM VARIABLES ACCORDING TO P"

Fix a set Ω called the *sample space model* of all *sample outcomes* $\omega \in \Omega$.
Subsets $A \subseteq \Omega$ are called *events*. English: "in the event that A occurs..."
An *observable* Θ is any deterministic function of the sample outcome

$$\Theta : \Omega \to \mathbb{R}$$

Θ has an easy job: if the experiment yields $\omega \in \Omega$, write down $\Theta(\omega) \in \mathbb{R}$.
Given any subset $B \subseteq \mathbb{R}$ in the codomain of Θ, have *preimage events*

$$\text{“}\Theta \in B\text{”} = \Theta^*(B) = \{\omega \in \Omega \ : \ \Theta(\omega) \in B\}$$

A *probability model* P on Ω predicts $A \subseteq \Omega$ will occur with probability

$$P(A) \in [0,1]_{\mathbb{R}}$$

P has a tough job: predict chances of all events before experiment is run!
For *all* Θ and *all* $B \subseteq \mathbb{R}$, P has to predict the probability that "$\Theta \in B$":

$$P(\Theta \in B) \in [0,1]_{\mathbb{R}}$$

This collection of P's predictions is the *distribution* of Θ *according to* P.
Once P is chosen, we say $\Theta : \Omega \to \mathbb{R}$ is a *random variable according to* P.
Measure theory: the art of making P's job easier so we can prove theorems!

States, trajectories, observables, and time averages

Let M be a set called the *state space model* of all possible *states* $u \in M$.
A *state trajectory* is a one-sided infinite sequence $(u_t)_{t=0}^{\infty}$ of states in M.
The *trajectory sample space* Ω_M is the set of all possible state trajectories

$$\Omega_M = \{(u_t)_{t=0}^{\infty} = (u_0, u_1, u_2, \ldots) \;:\; \text{all } u_t \in M\}$$

A *state observable* $\mathcal{O}$ is any deterministic function of state $\mathcal{O} : M \to \mathbb{R}$.
A state observable $\mathcal{O}$ *performed at time* τ is the function $\mathcal{O}_\tau : \Omega_M \to \mathbb{R}$

$$\mathcal{O}_\tau\big((u_t)_{t=0}^{\infty}\big) = \mathcal{O}(u_\tau)$$

A *trajectory observable* Θ is any deterministic function $\Theta : \Omega_M \to \mathbb{R}$.

Crucial: Even if $(u_t)_{t=0}^{\infty}$ isn't "random", can still define *time averages*

$$\frac{1}{T+1}\sum_{\tau=0}^{T}\mathcal{O}(u_\tau) = \frac{\mathcal{O}(u_0) + \mathcal{O}(u_1) + \mathcal{O}(u_2) + \cdots + \mathcal{O}(u_T)}{T+1}$$

of a state observable $\mathcal{O}$ *over partial trajectory* $(u_\tau)_{\tau=0}^{T}$ and study $T \to \infty$

Board: $M = \{a, b\}$ and $(u_0, u_1, u_2, \ldots) = (a, b, b, a, a, a, b, b, b, b, \ldots) \in \Omega_M$
and the state observable $\mathcal{O} : \{a, b\} \to \mathbb{R}$ with $\mathcal{O}(a) = 111$ and $\mathcal{O}(b) = 113$

Dynamical systems: state trajectories from recursions

Let M be a set. A *dynamical system* on M is any deterministic function

$$\varphi : M \to M.$$

Given any $u_0 \in M$, the *forward orbit of* u_0 *under* φ is the state trajectory

$$(u_0, u_1, u_2, u_3, \dots) \in \Omega_M$$

defined by iterating the recursion $u_{t+1} = \varphi(u_t)$ starting from $u_0 \in M$:

$$u_t = \varphi^t(u_0).$$

Since $u_0 \in M$ is arbitrary, get *initial-state-to-trajectory map* $\Phi : M \hookrightarrow \Omega_M$

$$\Phi(u_0) = (u_0, \varphi(u_0), \varphi(\varphi((u_0)), \varphi(\varphi(\varphi(u_0))), \dots)$$

<u>Remarkable</u>: On forward orbits, the value of any state observable $\mathcal{O}$ made at any time τ is just a function of the initial data $u_0 \in M$! Proof:

$$\mathcal{O}_\tau\Big((u_t)_{t=0}^{\infty}\Big) = \mathcal{O}(u_\tau) = \mathcal{O}(\varphi^\tau(u_0)) = (\mathcal{O} \circ \varphi^\tau)(u_0).$$

Board: $M = \{a, b\}$, consider $\varphi : \{a, b\} \to \{a, b\}$ with $\varphi(a) = b$ and $\varphi(b) = a$. The forward orbit of $u_0 = a$ under φ is $\Phi(a) =(a, b, a, b, a, b, \dots) \in \Omega_M$.

Dynamical systems in a random initial state $u_0 \in M$

Let $\varphi : M \to M$ be a dynamical system and $u_0 \in M$ an initial state.
Intuition: If $u_0 \in M$ is random, even though φ is deterministic,

- the forward orbit $(u_0, u_1, u_2, \dots) \in \Omega_M$ of u_0 under φ is random
- at any later time τ, the evolved state $u_\tau \in M$ is also random
- for any state observable $\mathcal{O}$, the numerical value $\mathcal{O}(u_\tau) \in \mathbb{R}$ is random
- most importantly, at any later time T, time averages are random!

$$\frac{1}{T+1}\sum_{\tau=0}^{T}\mathcal{O}(u_\tau) = \frac{\mathcal{O}(u_0) + \mathcal{O}(u_1) + \mathcal{O}(u_2) + \cdots + \mathcal{O}(u_T)}{T+1}$$

Precision: if $u_0 \in M$ is random according to a probability measure μ_0 on M,

- the forward orbit $(u_0, u_1, u_2, \dots) \in \Omega_M$ is random according to P for

  $$P = \Phi_*(\mu_0)$$

  the pushforward along the *initial-state-to-trajectory map* $\Phi : M \hookrightarrow \Omega_M$
- the evolved state $u_\tau \in M$ is random according to the measure μ_τ on M known as the time τ marginal distribution of P

Board: $M = \{a, b\}$, consider $\varphi : \{a, b\} \to \{a, b\}$ with $\varphi(a) = b$ and $\varphi(b) = a$.
If $u_0 \in \{a, b\}$ is chosen uniformly at random, what is $P(u_3 = b)$?

Dynamical systems in equilibrium: "stationary" $\neq$ "static"!

Summary: If $u_0 \in M$ is random according to μ_0 on M and $\varphi : M \to M$,

- the trajectory $(u_0, u_1, u_2, \ldots) \in \Omega_M$ is random according to P on Ω_M
- at any later time τ, the state $u_\tau \in M$ is random according to μ_τ on M

Equilibria: a probability model $\widetilde{\mu}$ on M is *stationary* or *invariant* under the dynamics generated by φ if when we choose $u_0 \in M$ at random using

$$\mu_0 \;=\; \widetilde{\mu}$$

and evolve u_0 by iterating φ, at all later times $\tau = 0, 1, 2, 3, \ldots$ we get

$$\mu_\tau \;=\; \widetilde{\mu}$$

Subtlety: if a dynamical system is in equilibrium, at any times τ_1 and τ_2, given a state observable $\mathcal{O} : M \to \mathbb{R}$,

- $\mathcal{O}(u_{\tau_1})$ and $\mathcal{O}(u_{\tau_2})$ are *identically distributed* random variables but
- $\mathcal{O}(u_{\tau_1})$ and $\mathcal{O}(u_{\tau_2})$ are typically *correlated* according to $\widetilde{P} = \Phi_* \widetilde{\mu}$

$$\mathrm{Corr}_{\tilde{P}}\Big[\mathcal{O}(u_{\tau_1})\ ,\ \mathcal{O}(u_{\tau_2})\Big] \neq 0$$

Board: how do we model iterated transposition dynamics in equilibrium? Can we detect any non-zero correlations? Why is this model not "static"?

Dynamical systems in equilibrium: Poincaré recurrence

Want to grow to love measure theory? Consider these 4 ingredients:

- A set M with elements $u \in M$: the *state space* of all possible *states*
- A function $\varphi : M \to M$ generating *deterministic dynamics* in M
- A probability model $\widetilde{\mu}$ for a random initial state $u_0 \in M$ that is stationary (invariant) under the dynamical system φ
- A σ-algebra $\mathcal{F}$ on M so φ is measurable and $\widetilde{\mu}$ is a measure.

Theorem 1 [H. Poincaré 1890] For any $U_0 \subseteq M$ "visible in $\widetilde{\mu}$-equilibrium"

$$0 < \widetilde{\mu}(U_0)$$

no matter how small, and any particular initial state $u_0 \in U_0$, the trajectory

$$(u_0, u_1, u_2, u_3, \ldots)$$

generated by φ from the initial state $u_0 \in U_0$ returns infinitely-often to U_0.

Why care? If you want to understand the out of equilibrium behavior of a deterministic dynamical system started in a particular initial state $u_0 \in M$, being able to construct invariant measures $\widetilde{\mu}$ for φ provides a means to prove that the dynamics you care about exhibits $\widetilde{\mu}$-*nearly perfect revivals*

Dynamical systems in equilibrium: Birkhoff thermalization

Want to grow to love measure theory? Consider these 4 ingredients:

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- A σ-algebra $\mathcal{F}$ on M so φ is measurable and $\widetilde{\mu}$ is a measure.

<u>Theorem 2</u> [G. D. Birkhoff 1931] Fix a state observable $\mathcal{O} : M \to \mathbb{R}$. Consider the event that an initial data $u_0 \in M$ in our dynamical system leads to a trajectory along which time averages of $\mathcal{O}$ converge to *some limit*:

$$\mathrm{Therm}_{\mathcal{O}}[\varphi] = \left\{ u_0 \in M \ : \ \exists \lim_{T \to \infty} \frac{1}{T+1} \sum_{\tau=0}^{T} \mathcal{O}(\varphi^{\tau}(u_0)) \ \in \mathbb{R} \right\}$$

If $\mathcal{O} : M \to \mathbb{R}$ viewed as a random variable according to $\widetilde{\mu}$ has a well-defined expected value $\mathbb{E}_{\tilde{\mu}}[\mathcal{O}]$ in $\widetilde{\mu}$-equilibrium then $\widetilde{\mu}$ predicts that the event above will occur with probability 1, namely, 100% of initial data "thermalize"!

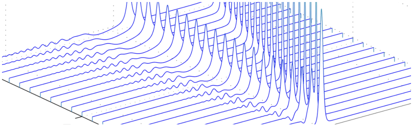
<u>Why care?</u> If you can prove that this system φ in equilibrium $\widetilde{\mu}$ is "*ergodic*" then for $\widetilde{\mu}$-almost every $u_0 \in M$, the limits above coincide with $\mathbb{E}_{\tilde{\mu}}[\mathcal{O}]$!

Hamiltonian PDEs out of equilibrium: joint with R. Chang

Problem: classical nonlinear waves of Benjamin (1967) and Ono (1975)

$$\begin{cases} \partial_t u = -u\partial_x u - \varepsilon_0 J[\partial_x^2]u \\ u(x,0) = u_0(x) \end{cases}$$

with $u_0(x)$ random according to some μ_0 on some Banach space M



- Nazarov-Sklyanin (2013): no quantum chaos in quantum version of this model
- M. (2019, *SIGMA*): evidence that no quantum chaos $\Rightarrow$ no classical chaos
- Gérard-Kappeler (2019): full analytic proof of no classical chaos
- M. (2023, *IMRN*): exploited lack of quantum chaos to prove quantum CLTs and discovered new simple combinatorial objects I called “ribbon paths”
- Cuenca-Dolega-M. (2025, *Annals of Probability*) bijections “ribbon paths” $\leftrightarrow$ complicated combinatorics in “free probability” yield new log-gas type CLTs
- Tvetkov (2023): exploits lack of classical chaos to construct new non-Gaussian invariant measures (stationary equilibria) - but are they ergodic? Open!
- Chang-Moll (2026): analytic proofs of classical and quantum CLTs from general Hamiltonian PDEs with Gaussian random initial data $u_0(x)$ which *don’t* require a lack of classical or quantum chaos, stability, or ergodicity!

Quantum ergodicity as irreversible wave function collapse

Revival of an irreversible wave function collapse after a measurement

$$|\Psi\rangle\langle\Psi|$$

taken during my Fall 2024 Reed Physics Colloquium in PHYS 123 titled

“Probability Theory and Quantum Revivals"