Probability Theory and Quantum Revivals

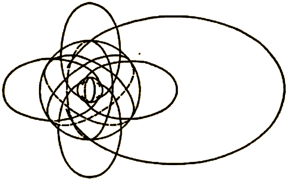

FIGURE I

Arrangement of electron orbits for the atom of neutral sodium. Orbits consisting partly of broken lines are circular orbits seen in perspective. The numbers and quantum relations of the orbits are as follows; inner shell, two 1_1 orbits; next shell, four 2_2 orbits and four 2_1 orbits; outer electron, one 3_1 orbit.

Figure 1 in Chapter 1 “The Laboratory Basis of Astrophysics” from Cecilia Payne’s Ph.D. Thesis *Stellar Atmospheres* (1925)

Will not discuss: quantization of classical periodic orbits, randomness in classical dynamics, or classical revivals

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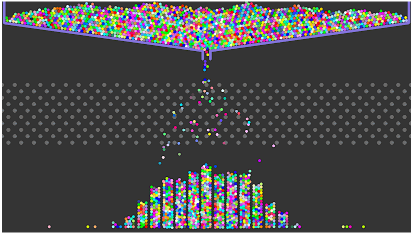
20 November 2024
Reed College Physics Colloquium

Randomness vs. Unpredictability: Chaos and Determinism

Problem: Predict that typical pile of particles ≈ normal Gaussian curve.



Tactic #1: model collisions using probability models and random variables as in our MATH 113, MATH 141, and MATH 391 *Probability.* Key assumption: outcomes of collisions are **independent.**

Tactic #2: model dynamics and collisions as in classical mechanics or our MATH 322 *Ordinary Differential Equations.* Key property: linear and nonlinear notions of **instability.**

Challenge: Justify that the CLT gives an effective description of dynamics.

⋆ Today: (1) Introduction to probability theory with dimensional analysis ⋆
(2) Quantum dynamics and revivals (3) Short-time asymptotics

Probability Models: "Random Variables" are Deterministic!

An **experiment** (such as a lab measurement, poll, or survey) is any reproducible procedure whose outcomes are possibly unpredictable.

A **sample space model** of an experiment is a set Ω with elements $w \in \Omega$ modeling possible outcomes of the experiment. **Events** are subsets $A \subseteq \Omega$.

A **random variable** X is any deterministic function $X : \Omega \to \mathbb{R}$ of $w \in \Omega$. Since outcomes $w \in \Omega$ are possibly unpredictable, so are random variables!

A **probability model** P predicts that the chance A will occur is "$P(A)$" while obeying Kolmogorov's axioms (1933): $0 \leq P(A) \leq 1$, $P(\Omega) = 1$, and

$$A \cap B = \emptyset \implies P(A \cup B) = P(A) + P(B).$$

⋆ Although X may have physical dimensions, P always **dimensionless**! ⋆

Intuition: P predicts chances of events *before* we run the experiment, but X has one job: *after* experiment, if outcome was w, write $X(w)$.

Compromise: Agree that random variables are "random according to P".

⋆ In applications, many P available to consult! Need criteria to select P ⋆

Densities of Random Variables: Continuous vs. Discrete

According to P, a random variable H has a density f_H on $\mathbb{R}$ if

$$P(a \leq H \leq b) = \int_a^b f_H(E)dE.$$

⋆ Assume: H and E both have physical dimensions of **energy.** ⋆
Since P dimensionless, f_H has physical dimensions of $\textbf{energy}^{-1}$
Careful: "f" will always mean energy density not "frequency"

- Gaussians: H is normal with mean $E_\star$ and variance σ^2 if

$$f_H(E) = \frac{1}{\sqrt{2\pi\sigma^2}}\; e^{-\frac{1}{2}\left(\frac{E-E_\star}{\sigma}\right)^2}$$

as in the experiment motivating the central limit theorem (CLT).

- Two-level system: if $P(H = E_1) = 0.3$ and $P(H = E_2) = 0.7$ then

$$\boxed{f_H(E) = 0.3\ \boldsymbol{\delta}(E - E_1) + 0.7\ \boldsymbol{\delta}(E - E_2)}$$

is a weighted sum of Dirac delta "functions" $\boldsymbol{\delta}$ at energies $E_1 \neq E_2$.

⋆ In 1925, Norbert Wiener traveled to work with Harald Bohr ⋆
to do research on Fourier transforms for irrational $\frac{E_1}{E_2} \in \mathbb{R} \setminus \mathbb{Q}$

Densities of Random Variables: Fourier Transforms

The Fourier transform of the density f_H of a random variable H is

$$\widehat{f}_H(t) = \mathbb{E}\left[e^{-\mathbf{i}(H/\hbar)t}\right] = \int_{-\infty}^{+\infty} e^{-\mathbf{i}(E/\hbar)t} f_H(E)dE$$

⋆ Planck's $\hbar$ has physical dimensions of **action** ⋆
so $\omega = E/\hbar$ has physical dimensions of **frequency!**

- Gaussians: if H is normal with mean $E_\star$ and variance σ^2 then

$$\widehat{f}_H(t) = e^{-\frac{1}{2}\sigma^2 t^2} \cdot e^{-\mathbf{i}\omega_\star t}$$

is a *damped* sinusoidal signal of angular frequency $\omega_\star = E_\star/\hbar$.

- Two-level system: if $P(H = E_1) = 0.3$ and $P(H = E_2) = 0.7$ then

$$\boxed{\widehat{f}_H(t) = 0.3\ e^{-\mathbf{i}\omega_1 t} + 0.7\ e^{-\mathbf{i}\omega_2 t}}$$

is a sum of *undamped* sinusoidal signals at frequencies $\omega_j = E_j/\hbar$.

⋆ If $\frac{E_1}{E_2} \in \mathbb{R} \setminus \mathbb{Q}$ irrational so frequencies ω_1, ω_2 are incommensurable ⋆
then $\widehat{f}_H(t)$ is what Harald Bohr called an *almost periodic function.*
This concept inspired Norbert Wiener's work in signal processing.

Example: Poisson Random Variables and Time Periodicity

Fix an angular frequency $\omega_\star$. Consider the discrete sequence of energies

$$E_n = \hbar\omega_\star\left(n + \frac{1}{2}\right)$$

indexed by dimensionless numbers $n \in \mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Note $E_0 = \frac{\hbar\omega_\star}{2}$.
Fix dimensionless $\alpha \in \mathbb{C}$. Consider the discrete random variable H with

$$f_H(E) = \sum_{n=0}^{\infty}\left(e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}\right)\boldsymbol{\delta}\Big(E - E_n\Big)$$

Goal #1 [on board]: The Fourier transform of the energy density f_H is

$$\widehat{f}_H(t) = e^{-\frac{\mathbf{i}\omega_\star t}{2}} \cdot e^{-|\alpha|^2\left(1-e^{-\mathbf{i}\omega_\star t}\right)}$$

Thus, the square modulus $S_H(t) = |\widehat{f}_H(t)|^2$ is the time-periodic function

$$\boxed{S_H(t) = e^{-2|\alpha|^2(1-\cos\omega_\star t)}}$$

Note: $H = \hbar\omega_\star\left(N + \frac{1}{2}\right)$ for N dimensionless Poisson of intensity $|\alpha|^2$.

⋆ Goal #2: relate our calculation of $S(t)$ to study of quantum revivals ⋆

Example: Cauchy Random Variables and Exponential Decay

Fix an energy E_{res} and an energy width Γ as e.g. in the recent textbook

- Dyatlov-Zworski *Mathematical Theory of Scattering Resonances* (2022).

Consider the continuous Cauchy random variable H with

$$f_H(E) = \frac{1}{\pi} \cdot \frac{\Gamma/2}{(\Gamma/2)^2 + (E - E_{\mathrm{res}})^2}$$

Exercise [not in talk]: The Fourier transform of the energy density f_H is

$$\widehat{f}_H(t) = e^{-(\Gamma/2)|t|} \cdot e^{-\mathbf{i}\omega_{\mathrm{res}} t}$$

a *damped* sinusoidal signal of angular frequency $\omega_{\mathrm{res}} = E_{\mathrm{res}}/\hbar$. Thus, the square modulus $S_H(t) = |\widehat{f}_H(t)|^2$ is the exponentially decaying function

$$\boxed{S_H(t) = e^{-\Gamma|t|}}$$

Note: this energy density is also called the *Breit-Wigner distribution*.

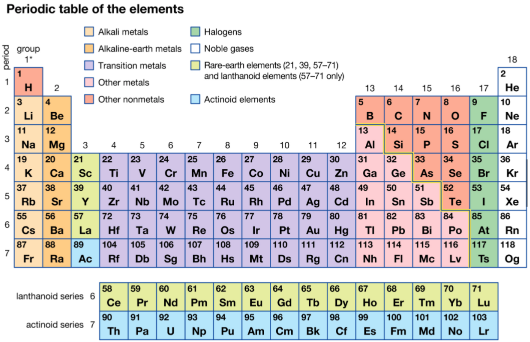
⋆ Riemann-Lebesgue: H absolutely continuous $\Rightarrow \lim_{t \to \pm\infty} S_H(t) = 0$ ⋆

Lévy: since H has heavy tails (no mean!), $S_H(t)$ is not analytic near $t = 0$

Glaring issue: the energy distribution is unbounded both above and below!

Quantum Phenomena: Quanta, Uncertainty, and Collapse

Quantization: “continuous” materials are made of **discrete quanta.**



Uncertainty: “quantum measurements” have **unpredictable outcomes.**

Collapse: “quantum measurements” cause **irreversible changes.**

⋆ To develop mathematical models of these quantum phenomena ⋆
helps to distinguish between **“random”** and **“unpredictable”**

Quantum Waves and Particles: Dispersion and Possibility

Problem: Develop mathematical models to predict quantum phenomena.

⋆ ⋆ ⋆ Born (1926): Schrödinger's Ψ is a dispersive wave of possibility ⋆ ⋆ ⋆

From Max Born's *Quantum Mechanics of* ***Collision*** *Processes* (1926)

> [Footnote 2 Page 2] *"Norbert Wiener of Cambridge, Massachusetts has graciously helped me with the mathematical details of this paper. I would like to express my thanks to him for that and acknowledge that I would not have reached my goal without him."*

⋆ Born attended Wiener's 1925 talk on uncertainty and Fourier transforms ⋆

Quantum Dynamics: Reversible and Irreversible

Model evolving quantum particle in (1+1)D by $\Psi : \mathbb{R}^1_q \times \mathbb{R}^1_t \to \mathbb{C}$ so

$$||\Psi(\cdot, t)||^2 = \int_{-\infty}^{+\infty} |\Psi(q,t)|^2 dq = 1$$

The Schrödinger equation: **in isolation (without measurements)**, the evolving state of a non-relativistic particle of mass m in a potential $V(q)$ solves the deterministic reversible partial differential equation (PDE)

$$\mathbf{i}\hbar \frac{\partial \Psi}{\partial t} = \widehat{H}\Psi$$

defined by the quantum Hamiltonian

$$\widehat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)$$

Hille-Yosida Theorem: for this autonomous homogeneous linear PDE,

$$\Psi(q,t) = e^{-\mathbf{i}(\widehat{H}/\hbar)t}\Psi(q,0)$$

⋆ Schrödinger's equation does not specify dynamics of **measurements** ⋆

⋆ Newton's equation does not specify dynamics of **collisions** ⋆

Quantum Measurements: Probability Models of Uncertainty

The Born Rule (1926): if a quantum particle is in an evolving state $\Psi(q,t)$, the outcome of measuring the **position** of this quantum particle at time t is a random variable $Q(t)$ with density

$$f_{Q(t)}(q) = |\Psi(q,t)|^2.$$

This means: the probability of observing the particle in $[a,b] \subset \mathbb{R}^1_q$ is

$$P(a \leq Q(t) \leq b) = \int_a^b |\Psi(q,t)|^2 dq$$

von Neumann (1932): If we measure the particle's **energy** H at time t, the outcome is a random variable H independent of t whose density f_H has

$$\widehat{f}_H(t) = \int_{-\infty}^{+\infty} \overline{\Psi(q,0)}\Psi(q,t)dq$$

Note: this $\widehat{f}_H(t)$ also called *fidelity* or *overlap* between $\Psi(0,t)$ and $\Psi(q,t)$.

$\star$ Interpret $S_H(t) = |\widehat{f}_H(t)|^2$ as **quantum survival probability** $\star$

Quantum Survival Probability: Revivals vs. Scattering

For evolving $\Psi(q,t)$, the quantum survival probability is

$$S_H(t) = |\widehat{f}_H(t)|^2 = \left| \int_{-\infty}^{+\infty} \overline{\Psi(q,0)}\Psi(q,t)dq \right|^2$$

a measure of the overlap between the initial and evolved states.

H.Bohr - Wiener: if the quantum energy H measured in the initial state $\Psi(q,0)$ is a discrete random variable supported on a finite or countably infinite set of energies E_n then the quantum survival probability is always an almost periodic function of time t

- In this case, the quantum particle undergoes almost perfect revivals

Riemann - Lebesgue: if the quantum energy H measured in the initial state $\Psi(q,0)$ is a absolutely continuous random variable then the quantum survival probability **decays to zero as** $t \to \infty$

- Beyond the scope of the talk: escape probabilities and **scattering**.

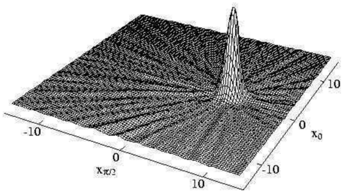
⋆ Now do special example with perfect quantum revivals ⋆

Example: Quantum Harmonic Oscillator in Coherent State

Fix dimensionless $\alpha \in \mathbb{C}$, mass m, and preferred oscillation frequency ω_c.
As special initial states, consider **coherent states** $\Psi(q,0) = \Upsilon_\alpha(q)$ with

$$\left|\Upsilon_\alpha(q)\right|^2 = \sqrt{\frac{m\omega_c}{\pi\hbar}}\; e^{-\frac{m\omega_c}{\hbar}(q-q_\alpha)^2}$$

Υ_α is a Gaussian wavepacket localized in space at $q_\alpha = \sqrt{\frac{2\hbar}{m\omega_c}}\mathrm{Re}[\alpha]$.



☐ Breitenbach-Schiller-Mlynek
"Measurement of the quantum states of squeezed light." *Nature* 387 (1997) 471-5.

<u>Exercise:</u> nice dynamics of $\Psi(q,t)$ in case of *quantum harmonic oscillator*

$$\widehat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + \frac{1}{2}m\omega_\star^2 q^2$$

Careful: state evolves through *squeezed coherent states* if preferred $\omega_c \neq \omega_\star$

⋆ Omit: relate $\alpha \in \mathbb{C}$ to underlying classical phase space $\mathbb{R}^1_q \times \mathbb{R}^1_p$ ⋆

Finale: Quantum Revivals of Coherent States

Goal #2: If we consider the *quantum harmonic oscillator* defined by

$$\widehat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + \frac{1}{2}m\omega_\star^2 q^2$$

and choose an initial coherent state at preferred frequency $\omega_c = \omega_\star$

$$\Psi(q,0) = \Upsilon_\alpha(q)$$

the quantum survival probability can be computed exactly as

$$\boxed{S_H(t) = e^{-2|\alpha|^2(1-\cos\omega_\star t)}}$$

so Υ_α undergoes perfect **quantum revivals** every

$$\tau_{\text{rev}} = \frac{2\pi}{\omega_\star}$$

Why? Once you show the famous statement that

The number distribution N of Υ_α is Poisson of intensity $\lvert\alpha\rvert^2$

the calculation follows by our work in Goal #1.

⋆ Omit: similar story for non-quadratic potentials $V(q)$ ⋆

Contrast: Quantum Tunneling and Exponential Decay

If a quantum system actually had a Breit-Wigner distribution of energies, by the unsolved Exercise above, could calculate the survival probability

$$\boxed{S_H(t) = e^{-\Gamma|t|}}$$

Early triumph of mathematics touched on in this talk: Gamow (1928) and Gurney-Condon (1928) matching of the new wave mechanics for Ψ to estimate the decay rates in $\boldsymbol{\alpha}$-emission.

> *"It has hitherto been necessary to postulate some special arbitrary 'instability' of the nucleus, but in the following note, it is pointed out that disintegration is a natural consequence of the laws of quantum mechanics without any special hypothesis. Much has been written of the explosive violence with which the* $\boldsymbol{\alpha}$*-particle is hurled from its place in the nucleus. But from the process pictured above, one would rather say that the* $\boldsymbol{\alpha}$*-particle almost slips away unnoticed."*

☐ Ronald Gurney and Edward Condon (*Nature* 1928)
"Wave Mechanics and Radioactive Disintegration"

⋆ ⋆ ⋆ ⋆ ⋆ ⋆ **independence** $\Longleftrightarrow$ **instabilities** ⋆ ⋆ ⋆ ⋆ ⋆ ⋆

Asymptotics: A Dynamical Proof of the Poisson CLT

Fall 2024 MATH 391 *Probability* Midterm 1 Problem 4.4:

- In the limit $|\alpha|^2 \to \infty$ of high intensity $|\alpha|^2$, a Poisson random variable

$$P(N = n) = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}$$

is approximately normal. This is secretly a special case of the CLT!

Dynamical origin: at short times t, the quantum survival probability

$$\boxed{S_H(t) = e^{-2|\alpha|^2(1-\cos \omega_\star t)}}$$

seems to exhibit Gaussian decay via Taylor approximation $\cos\theta \approx 1 - \frac{1}{2}\theta^2$:

$$S_H(t) \approx e^{-|\alpha|^2 \omega_\star^2 t^2}$$

⋆ Details with **Robert Chang**: separation of scales in semiclassical limit ⋆

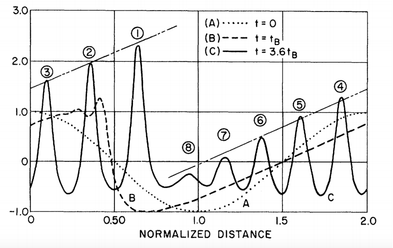
⋆ Moral: if you only probe short times, you have **no idea** ⋆
whether or not you'll see quantum revivals or scattering

Research: Soliton Quantization and Quantum Revivals

My research: use “quantization” and “uncertainty” of “collisions” to study

1. **Combinatorics:** enumeration of graphs on non-oriented real surfaces
2. **Probability:** correlations and limits in models of random partitions
3. **Dynamics:** quantization of **solitons** and nonlinear dispersive waves

via special Hamiltonian QFTs which exhibit quantum effects but no chaos.



☐ Zabusky-Kruskal “Interaction of **‘Solitons’** in a Collisionless Plasma and the **Recurrence of Initial States**.” *Physical Review Letters* 15 (1965) 240-3.

Zabusky-Kruskal discover famous classical revivals in

$$\begin{cases} \partial_t u + u\partial_x u = \partial_{xxx} u \\ u(x,0) = \cos x \end{cases}$$

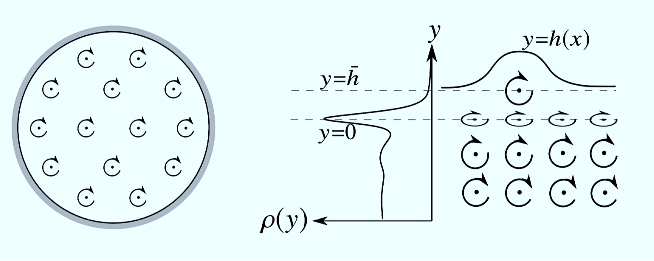
to explain similar phenomena in the Fermi-Pasta-Ulam-Tsingou experiment.

⋆ Future goals: quantum revivals in quantizations of such classical PDEs ⋆

At Reed: Soliton Quantization via Quantum Mechanics

*"When field-theory "constructivists" proved the existence of quantum $\lambda\phi^4$ theory in $(1+1)$-dimensions, they were correct, but missed the entire **quantum soliton** phenomenon, which is the only physically interesting feature of that model."*

☐ R. Jackiw "My Encounters as a Physicist with Mathematics" (1996)



☐ Wiegmann "Nonlinear hydrodynamics and fractionally **quantized solitons** at the fractional quantum Hall edge." *Physical Review Letters* 108 (2012).

⋆ Thank You! ⋆